1 Problem 1

1.1

The voltage across the capacitor (V) increases with time as follows:

$$V = V_i(1 - e^{-\frac{t}{RC}})$$

but when it reaches the firing voltage V_f, it immediately drops to a negligible value $V = 0$ and the process repeats.

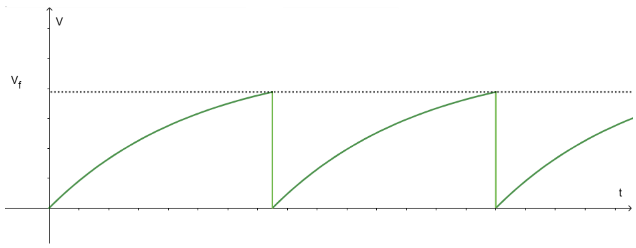

Figure 1: Output waveform $V(t)$

1.2

The output voltage in each period is:

$$V = V_i(1 - e^{-\frac{t}{RC}})$$
$$= V_i(1 - (1 - \frac{t}{RC} + \frac{1}{2!}(\frac{t}{RC})^2 - \frac{1}{3!}(\frac{t}{RC})^3 + \cdots))$$

for the voltage to be almost linear, the value of RC must be high such that the higher order terms containing $\frac{1}{RC}$ can be neglected. In that case the voltage is

$$V = V_i\frac{t}{RC}$$

1.3

Each cycle starts with $V = 0$ and ends with $V = V_f$. And hence time period T is such that:

$$V_f = V_i\frac{T}{RC}$$
$$T = \frac{V_f}{V_i}RC$$

1.4

To vary timeperiod only, R must be varied, because if SG and hence V_f is varied the amplitude is also varied.

1.5

To vary amplitude only, SG and R must be varied.

2 Problem 2

This problem has symmetry; all snails are symmetrically positioned and move with constant speed and thus as they move their position makes equilateral triangle of decreasing side length and hence they finally meet at centroid of the original equilateral triangle.

Relative velocity of first snail towards second one, along the side of triangle is :

$$V_r = V_A + V_B\cos(60)$$
$$= 5 + \frac{1}{2} * 5$$
$$= \frac{15}{2}$$

The other component of relative velocity, along the perpendicular to the side of triangle, makes the first snail seem to rotate about the second snail and thus not contributing to decrease the distance between them. Since these speeds are same at each instant. Time taken to cover the distance 60 cm is

$$T = \frac{60}{\frac{15}{2}}$$

$$T = 8 \text{ minutes}$$

and the total distance covered by each snail during that time is

$$S = 5 * 8 = 40cm$$

This problem can be solved by other methods too. Any valid method, producing the correct answer is accepted.

3 Problem 3

Let total length of carpet be L, the length of carpet to the right of fixed end be l_1 and, half of the total length of the carpet to the left be l_2. Considering the fixed end as origin and assuming unit mass per length, the position of CG is: (Assuming a constant mass per unit length $= \lambda$, doesn't affect the results)

$$x_{CG} = \frac{2l_1(-\frac{l_1}{2}) + l_2\frac{l_2}{2}}{L}$$

Differentiation wrt time

$$\dot{x}_{CG} = \frac{-2l_1\dot{l_1} + l_2\dot{l_2}}{L}$$

It can trivially shown that $l_2 = vt$, $2l_1 = L - l_2 = L - vt$ and $2\dot{l_1} = -\dot{l_2}$, so the above equation becomes:

$$\begin{aligned} L\dot{x}_{CG} &= -2l_1\frac{-\dot{l_2}}{2} + l_2\dot{l_2} \\ &= (l_1 + l_2)\dot{l_2} \\ &= (\frac{L - vt}{2} + vt)v \\ &= \frac{v}{2}(L + vt) \\ \dot{x}_{CG} &= \frac{v}{2}(1 + \frac{vt}{L}) \end{aligned}$$

This is the velocity of the center of mass.

4 Problem 4

Probability that a person is detected $p = 0.8$

4.1

So, probability that the sensor detects 8 men $= C(10, 8)p^8(1 - p)^2$

4.2

If the sensor detects 8 men, then the probability to detecting other 2 men is independent of the former probability. So, the probability that it detects last two men is p^2

4.3

Probability of Detecting k people among N is

$$P(k) = C(N,k)p^k(1-p)^{N-k}$$

for $p = 0$, $P(k) = 0$ except for $P(0) = 1$ hence, the sensor most likely detect no person at all .
for $p = 1$, the sensor detect all people.
and thus,

$$\frac{P(k+1)}{P(k)} = \frac{N-k}{k+1}\frac{p}{1-p}$$

(unless p = 0 or p = 1)
Now,

$$\begin{aligned} P(k+1) < P(k) &\implies k > (N+1)p - 1 \\ P(k+1) = P(k) &\implies k = (N+1)p - 1 \\ P(k+1) > P(k) &\implies k < (N+1)p - 1 + 1 \end{aligned}$$

So, if $k_0 = (N+1)p - 1$ is an integer $P(k)$ is maximum for $k = k_0$ and $k = k_0 + 1$, else P increases for $k < k_0$ and decreases for $k > k_0$; Thus $P(k)$ is maximum for largest integer less than k_0 i.e. $k = \lfloor (N+1)p \rfloor$

5 Problem 5

5.1

From geometry (or other methods) the area of the polygons (quadilaterals) can be obtained.
For M_1:

$$A_1 = 1$$
$$A_1' = 7$$
$$\frac{A_1'}{A_1} = 7$$

For M_2:

$$A_2 = 4$$
$$A_2' = 1$$
$$\frac{A_2'}{A_1} = 1/4$$

5.2

$$|M_1| = 7$$
$$|M_2| = 1/4$$

The ratio of areas of transformed polygon to original polygon is equal to the determinant of the Matrix.

5.3

The top face of the cube is displaced by $\delta h = 0.01cm$ because :

$$M_3 . \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0 \\ 1 \end{bmatrix}$$

Shear force F is given by:

$$F = \frac{GA\delta h}{h} = 75\text{kN}$$

5.4

When a cuboid $(l * b * h)$ is placed as above, the top face is displaced by

$$\delta h = m_{13} * h$$

hence,

$$m_{13} = \frac{\delta h}{h} = \frac{F}{lbG}$$

and the transformation matrix is:

$$M = \begin{bmatrix} 1 & 0 & \frac{F}{lbG} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$