Maths-Physics Olympiad

January 19, 2020

Solution

Moment of Intertia of a uniform rod of mass M rotating about an axis perpendicular to its length and passing through its centre is:

$$I = \frac{1}{12}ML^2 \tag{1}$$

You may need the following integral:

$$\int_0^\infty x^{2n}e^{-\alpha x^2}dx = \frac{1.3.5\ldots(2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}} \tag{2}$$

Spring

The force on the combined mass $M + m$ at any position z is:

$$F = mg - k(z - z_0) \tag{3}$$
$$(M+m)\frac{d^2z}{dt^2} = -k(z - z_0 - \frac{mg}{k}) \tag{4}$$
$$\frac{d^2z}{dt^2} = -\frac{k}{M+m}(z - z_0 - \frac{mg}{k}) \tag{5}$$
$$\frac{d^2z}{dt^2} = -\omega^2(z - z') \tag{6}$$
$$\tag{7}$$

This is the differential equation of SHM. Whose general solution is:

$$z = Asin(\omega t) + Bcos(\omega t) \tag{8}$$

Using the initial conditions

1. $z = z_0$ at $t = 0$
2. $z' = 0$ at $t = 0$

We get the following solution:

$$z = z_0 + \frac{mg}{k}(1 - \cos\frac{mg}{k}t) \tag{9}$$

Alternatively

This system can be thought of as a spring with mass $M + m$ that was first at its initial position $z' = z_0 + \frac{mg}{k}$ but was moved up to z_0 by a force that was then removed at $t = 0$. This results in a SHM (Simple Harmonic Motion) with amplitude of $\frac{mg}{k}$ about z'. Whose equation can be written by inspection as :

$$z = z_0 + \frac{mg}{k}(1 - \cos\frac{mg}{k}t) \tag{10}$$

A rotating rod

For perfectly elastic collision, energy is conserved. So,

$$\frac{1}{2}Mv^2 + \frac{1}{2}Iw^2 = \frac{1}{2}mv'^2 \tag{11}$$

where v' is the velocity of mass m after impact.
Conservation of Linear momentum along the direction of v entails:

$$Mv = mv' \tag{12}$$

And conservation of angular momentum with origin at the centre of the rod results:

$$Iw = mv'\frac{l}{2} \tag{13}$$

Now using $I = \frac{1}{12}Ml^2$ and equations 11, 12, 13 gives the following ratio:

$$\frac{M}{m} = 4 \tag{14}$$

The Perfect Turn

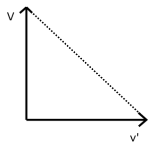
1. While running, the normal force exerted by the boy fluctuates with time. But since the initial and final velocity in up direction is zero. There is no net acceleration along upward direction. Also, the weight is constantly acting in vertically downward direction. So, although

$$N \neq W \tag{15}$$

but

$$\text{average } N = \text{average } W \tag{16}$$
$$N_{av} = W \tag{17}$$

2. The acceleration of the boy can never exceedd $a = \frac{F}{m} = \frac{\mu N}{m} = \frac{\mu W}{m} = \mu g$. If it does, he will slip. His initial velocity is along north v and final velocity is along east v'. So, for minimal time of turn, what should be his acceleration? (constant, or changing, and if changing how so?)



See figure, the speed at which the tip of velocity vector v reaches v' is the acceleration. In the optimal case, the tip of the velocity vector must trace a straight line. So, the acceleration must be constant and time equal to:

$$t = \frac{|\boldsymbol{v'} - \boldsymbol{v}|}{\mu g} = \frac{\sqrt{2}v}{\mu g} \tag{18}$$

3. Since the acceleration is constant, the trajectory during the turn will be parabola. Which can now be easily obtained assuming any origin and coordinate axes.

Probability Distribution Function

1. Since, probability ($f(v)\,\mathrm{d}v$) is unitless and $\mathrm{d}v$ has the unit of m/s the unit of probability distribution function is s/m.

2. Since, all the particles have some definite velocity, the integral $\int f(v)\,\mathrm{d}v$ over all v representing the total probability of the particles having some velocity, is 1.

3. The number $Nf(v)\,\mathrm{d}v$ signifies the **most probable** number of particles having the velocity between v and $v + \mathrm{d}v$.

4. Given that average value of $g(v)$ is $\int g(v)f(v)\,\mathrm{d}v$, the average value of v is:

$$\int vf(v)\,\mathrm{d}v \tag{19}$$

5. and the expression of v_{rms} is

$$v_{rms} = \sqrt{\text{average of } v^2} \tag{20}$$
$$= \sqrt{\int v^2 f(v)\,\mathrm{d}v} \tag{21}$$

6. Using

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{-mv^2/2kT} \tag{22}$$

in eqn 20 and using the integral provided:

$$\int_0^\infty x^{2n} e^{-\alpha x^2}\,\mathrm{d}x = \frac{1.3.5\ldots(2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}} \tag{23}$$

we can easily obtain that:

$$v_{rms} = \sqrt{\frac{3kT}{m}} \tag{24}$$

7. Using $\frac{k}{m} = \frac{R}{M}$ we get:

$$v_{rms} = \sqrt{\frac{3RT}{M}} \tag{25}$$

Eight Odd Squares

Using Lagrange's Four-Square Theorem:

$$8N = 8i^2 + 8j^2 + 8k^2 + 8l^2 \tag{26}$$

where i, j, k, l are integers. Also,

$$8i^2 = (4i^2 + 4i) + (4i^2 - 4i) \tag{27}$$

$$= (2i+1)^2 + (2i-1)^2 - 2 \tag{28}$$

is expressed as sum of two odd squares minus 2. So,

$$\begin{aligned} 8N = (2i+1)^2 + (2i-1)^2 + (2j+1)^2 + (2j-1)^2 \\ (2k+1)^2 + (2k-1)^2 + (2l+1)^2 + (2l-1)^2 - 8 \\ 8(N+1) = \text{Sum of eight odd squares} \end{aligned}$$

Since any positive multiple of 8 can be expressed as $8(N+1)$ which can in turn be expressed as sum of eight odd square, the result required is proved. QED.

Entropy

1. Given function H whose extremum is to be found is:

$$H = \sum_{i=1}^{n} -p_i \log(p_i) \tag{29}$$

and the constraint is:

$$\sum_{i=1}^{n} p_i = 1 \tag{30}$$

Lets first generalize the constaint to be :

$$\sum_{i=1}^{n} p_i = T \tag{31}$$

and lets denote the function H to be maximized under this contraint as $H_n^T = \sum_{i=1}^{n} -p_i \log(p_i)$ where n number of terms p_i.
For $n = 1$ and any T:
The constraint results to $p_1 = T$, hence the only value of $H_n^T = H_1^T = -Tlog(T)$ is the extremum.

Assume For $n = k$
Assuming that the function H_n^T is maximized by $p_1 = p_2 = ... = p_k = T/k$ for $n = k$ for any T, the maxima of H_k^T is:

$$H_k^T = -k(T/k)\log(T/k) = -T\log(T/k) \tag{32}$$

Prove for $n = k + 1$
Now we need to prove that the function is also maximized for $n = k + 1$ for any T. So, for $n = k + 1$:

$$H_{k+1}^T = \sum_{i=1}^{k+1}(-p_i \log p_i) \tag{33}$$

$$= -p_1 \log(p_1) + \sum_{i=2}^{k+1}(-p_i \log p_i) \tag{34}$$

The second sum is the function $H_n^{T'}$ for number of terms $n = k$ and the constraint value T' equal to $\sum_{i=2}^{k+1} p_i = \sum_{i=1}^{k+1} p_i - p_i = T - p_i$.

$$H_{k+1}^T = -p_1 \log(p_1) + H_k^{T-p_1} \tag{35}$$

The maxima of the second term is $-(T - p_1)\log(\frac{T-p_1}{k})$ at $p_2 = p_3 = ... = p_{k+1} = \frac{T-p_1}{k}$ by assumption 32. So, equating the derivative of 35 wrt p_1 to 0. We get:

$$\frac{\mathrm{d}}{\mathrm{d}p_i} H_{k+1}^T = \frac{\mathrm{d}}{\mathrm{d}p_i}(-p_1 \log p_1 - (T - p_1)\log(\frac{T - p_1}{k})) \tag{36}$$

$$= -(\log p_i + 1) + (\log(\frac{T - p_1}{k}) + 1) \tag{37}$$

$$\log p_1 + 1 = \log(\frac{T - p_1}{k}) + 1 \tag{38}$$

$$p_1 = \frac{T - p_1}{k} \tag{39}$$

$$p_1(k + 1) = T \tag{40}$$

$$p_1 = \frac{T}{k + 1} \tag{41}$$

So,

$$p_1 = p_2 = ...p_{k+1} = \frac{T}{k + 1} \tag{42}$$

extremizes the function. Also, computing the second derivative of H_{k+1}^T from second step of 36 gives:

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 p_i} H_{k+1}^T = -\frac{T}{T - p_1} \qquad \leq 0 \tag{43}$$

So, relation 42 indeed maximizes the function H_{k+1}^T. Since, the condition for maxima is true for $n = 1$ and also for $n = k + 1$ when it is true for $n = k$, it is true for any n by Mathematical Induction. Also, setting $T = 1$ proves the condition of maxima for our original function 29.

2. (a) Since the events A_i and A_k are mutually exclusive, the conditional probability $P(A_i|A_k) = 0$

(b) As $P(A_i|A_k) = 0$ for $i \neq k$ and $\log P(A_i|A_k) = 0$ for $i = k$.

$$H = \sum -P(A_i|A_k) \log P(A_i|A_k) \qquad = 0$$

(c) When the events are not mutually exclusive, occurence of one doesn't completely eliminated the possibility of occurence of other and hence some uncertainity remains in the experiment which can result to non zero Infromation Entropy.

Alternative proof

Using Lagrange's Multiplier strategy, the function to be maximized H under the constraint can be expressed as a function F that needs to be maximized without any constraint.

$$F = H + \lambda(\text{Constraint}) \tag{44}$$

$$= \sum(-p_i \log(p_i)) + \lambda(\sum p_i - 1) \tag{45}$$

Equating partial derivative of F with respect to p_k to zero gives:

$$\frac{\mathrm{d}F}{\mathrm{d}p_k} = -(\log p_k + 1) + \lambda p_k \tag{46}$$

$$0 = -(\log p_k + 1) + \lambda p_k \tag{47}$$

$$\lambda = \frac{\log p_k + 1}{p_k} \tag{48}$$

Equating λ obtained by partial derivative wrt p_1 with the λ from p_2 and so on, we get:

$$p_1 = p_2 = ... = p_n = \frac{1}{n} \tag{49}$$

QED.