Maths-Physics Olympiad

January 29, 2019

Voltage across a charging capacitor is:

$$V = V_0(1 - e^{-t/RC})$$

where V_0 is EMF of battery.

For Steel

1. Modulus of Rigidity (Shear Modulus) $G = 75$ GPa
2. Modulus of Elasticity (Young's Modulus) $Y = E = 200$ GPa
3. Bulk Modulus $K = 163$ GPa

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Problem 1 [7]

A sawtooth voltage waveform V can be obtained across the capacitor C in (fig: 1), R is a variable resistor, V_i is an ideal battery, and SG is a spark gap consisting of two electrodes with an adjustable distance between them. When the voltage across the electrodes exceeds the firing voltage V_f , the air between the electrodes breaks down, hence the gap becomes a short circuit and remains so until the voltage across the gap becomes very small. (Here $V_f < V_i$)

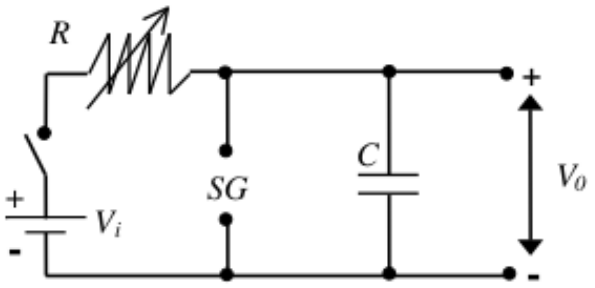


Figure 1: Sawtooth Generator Circuit

1. Express V as function of time and draw the voltage waveform V versus time t, after the switch is closed. [2]
2. What condition must be satisfied in order to have an almost linearly varying sawtooth voltage waveform V_0 ? [2]
3. Provided that this condition is satisfied, derive a simplified expression for the period T of the waveform. [1]
4. What should you vary (R and/or SG) to change the period only? [1]
5. What should you vary (R and/or SG) to change the amplitude only? [1]

Problem 2 [5]

Three small snails are each at the vertex of an equilateral triangle of side 60 cm. The first snail sets out towards the second, the second towards the third and the third towards the first with a constant speed of 5cm/min. As they go they always head towards their respective target snail. How much time has elapsed and what distance the snails conver by the time they meet?

Problem 3 [4.5]

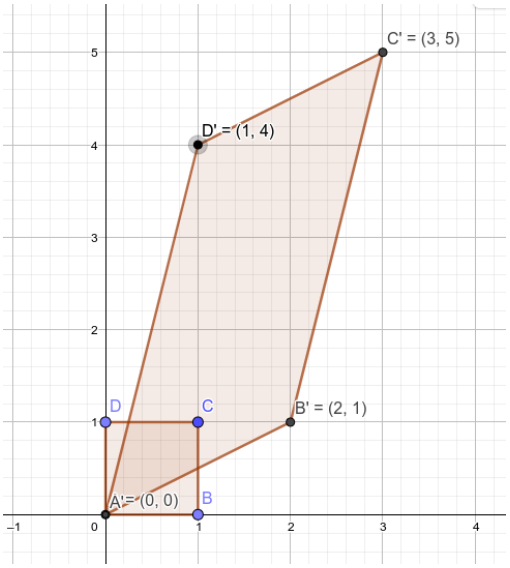
A carpet, folded exactly in half, is fixed at one end. The free end, overlying on the fixed end, is pulled with a constant velocity v. Find the velocity of center of mass of carpet as a function of time t and velocity of pull v.

Figure 2: Carpet: Lower end fixed, upper end being pulled at constant velocity v

Problem 4 [5.5]

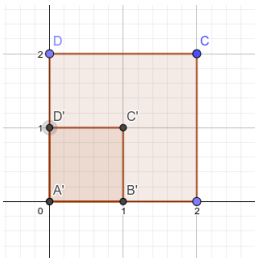
A sensor is used to detect passage of people in a gate. It can detect a person with 80% accuracy. When 10 men enter the gate:

1. What is the probability that the sensor detect 8 men? [1]
2. If the sensor detects 1st 8 men, what is the probability that it also detects other 2 men? [1]



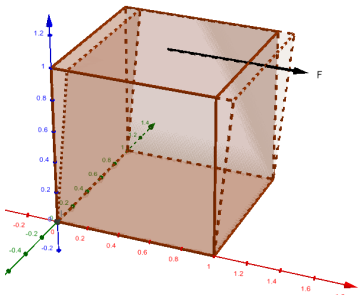
(a)

$$M_1 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$



(b)

$$M_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$



(c)

$$M_3 = \begin{bmatrix} 1 & 0 & 0.01 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 3: Transformation using matrices

3. If the sensor detects people with an accuracy of p. Among N persons, how many people will the sensor most likely detect? (Hint: take the ratio $\frac{P(k+1)}{P(k)}$, where $P(k)$ is the probability that the sensor detects k people among N people and find k for which $P(k)$ is maximum) [3.5]

4. Use the above result for the case of $N = 10$ and $p = 80\% = 0.8$ i.e. how many people will the sensor most likely detect? [0.5]

Problem 5 [7.5]

Matrices can be used to represent transformations (rotation, scaling, shear, etc). Matrix M_1 transforms a unit square $ABCD$ to $A'B'C'D'$ and the matrix M_2 transforms the square $ABCD$ to $A'B'C'D'$ as shown in (fig: 3a) and (fig: 3b).

For example the vetex C is transformed by M_1 to C' as: $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

1. Find the ratio of area of transformed polygon to the original polygon for both transformations. [1]

2. Find the determinants $|M_1|$ and $|M_2|$ of the transformation matrices and observe the relation between the ratio and the determinant. [0.5]

3. Prove the relation you obtained for any transformation represented by 2 by 2 matrix applied on an unit square. (Transform the vertices of the unit square and compute the area of resulting quadilateral) [2]

A unit cube of steel (1 cu. cm), placed with a vertex at origin and the sides aligned along the axis, is transformed under the application of shear force F. The transformation is represented by the matrix M_3 (fig: 3c).

1. Find the force F. [2]

2. If a steel cuboid (with dimensions $l * b * h$ in x, y and z axes respectively) is placed similarly to the unit cube above and a shear force F' is applied on its top face along x-axis, find the transformation matrix. [2]