Maths-Physics Olympiad

January 19, 2020

1. Total Points: 30
2. Total time: 2 hours
3. Fractional marking will be done, so don't forget to metion relevant facts even if you can't completely solve a problem.
4. You can use the fact given in a question to solve another question/sub-question even if you can't prove the original fact.

Moment of Intertia of a uniform rod of mass M rotating about an axis perpendicular to its length and passing through its centre is:

$$I = \frac{1}{12}ML^2 \tag{1}$$

You may need the following integral:

$$\int_0^\infty x^{2n}e^{-\alpha x^2}dx = \frac{1.3.5\ldots(2n-1)}{2^{n+1}\alpha^n}\sqrt{\frac{\pi}{\alpha}} \tag{2}$$

Spring [4]

Consider a mass M suspended on a spring with spring constant k. The system is initially at equilibrium with mass M at position $z = z_o$. A small mass m just above the mass M is released at time $t = 0$ as shown in (fig: 1). The collision is perfectly inelastic and the small mass remains attached to the bigger mass M. Find the position of mass M at time t. i.e. Find $z(t)$.

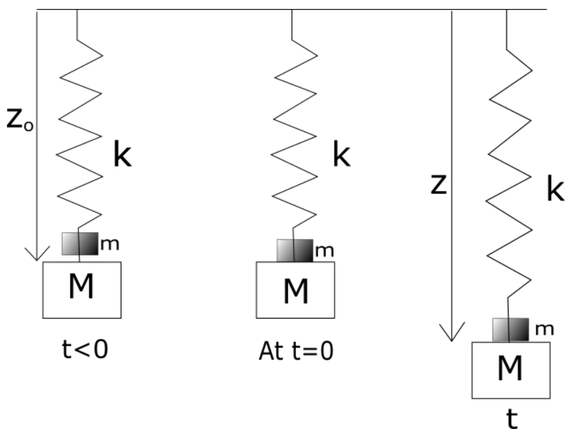

Figure 1: Small mass dropped on a spring

A rotating rod [4]

A straight, uniform rod of mass M is sliding in ice with a linear center of mass velocity v, while also rotating with an angular velocity of ω, as shown in (fig: 2). At an instant when its velocity is perpendicular to its orientation, the rod collides perfectly elastically with a ball of mass m. What should be the ratio of M and m, i.e. $\frac{M}{m}$, in order for the rod to come to rest after the impact?

Consider using conservation laws

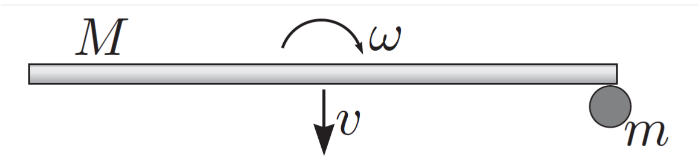

Figure 2: Rotating Rod colliding with a ball

The Perfect Turn [5]

A boy is running north, on the smooth ice cover of a large frozen lake, with a speed v. The coefficient of friction (both kinetic and static) between the sole of his shoe and the ice is μ. For the sake of simplicity, assume that the normal force he exerts on the ice, which in reality changes with time, can be substituted by its average value.

1. The average normal force is equal to the weight of the boy. Why? [**1.0**]
2. What is the minimal time that he needs to change direction, so that he is running east with the same speed v? [**3**]
3. Find the boy's trajectory during the turn in this optimal case. [**1.0**]

Probability Distribution Function [5.5]

Probability distribution function of a varibale is a function that describes the relative likelihood that the variable will take on a specific value. One well known example of such probability distribution function is the Maxwell-Boltzmann distribution function. This function $f(v)$ gives the distribution of molecular speeds in a gas container. The probability that a randomly chosen molecule will have a speed between v and $v + \mathrm{d}v$ is given by $f(v)\,\mathrm{d}v$.

1. Determine the unit of the distribution function. [**0.5**]

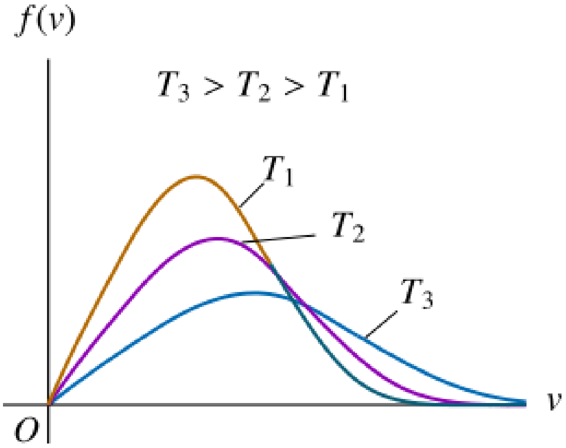

Figure 3: Maxwell–Boltzmann distribution

2. What is the value of the integral of $\int f(v)\,\mathrm{d}v$ over all v? **[0.5]**
3. If the container has N number of gas molecules, what does $Nf(v)\,\mathrm{d}v$ signify? **[1]**
4. The average value of a function (say function of x, $g(x)$) under the probability distribution ($f(x)$) of x is given by the integral $\int g(x)f(x))\,\mathrm{d}x$ integrated over all possible values of x.
   In our case, the average value of any function of speed $g(v)$ is $\int g(v)f(v)\,\mathrm{d}v$. Write the expression for average speed v_{av}. You may leave the final answer in integral form. **[0.5]**
5. Similarly write the expression for root mean square speed v_{rms}. (Root mean square speed is the square root of the mean square speed (i.e. mean of v^2)). **[0.5]**
6. The actual expression of the distribution function $f(v)$ is

$$f(v) = 4\pi\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} v^2 e^{-mv^2/2kT} \tag{3}$$

   Here m is the mass of a molecule, k is the Boltzmann constant and T is the temperature in Kelvin. Calculate the value of v_{rms}. **[2]**
7. Simplify the previous expression using $\frac{k}{m} = \frac{R}{M}$ in the previous answer. You may see a familiar result from your high school thermodynamics classes. **[0.5]**

Eight Odd Squares [4]

Lagrange's Four-Square Theorem states that every positive integer can be written as the sum of at most four integer squares (i.e. $N = i^2 + j^2 + k^2 + l^2$). For example: $31 = 5^2 + 2^2 + 1^2 + 1^2$.
Given this theorem prove that any positive multiple of 8 can be written as the sum of eight odd suqares (An odd square means square of an odd number).

Consider $8i^2 = (4i^2 + 4i) + (4i^2 - 4i)$

Entropy [7.5]

The probability $P(A)$ of an event A can be interpreted as a measure of our uncertainty about the occurrence or nonoccurrence of A in a single performance of the underlying experiment S. If $P(A) \approx 0.999$, then we are almost certain that A will occur; if $P(A) = 0.1$, then we are reasonably certain that A will not occur; our uncertainty is maximum if $P(A) = 0.5$.
If we consider an experiment S with n mutually exclusive exhaustive events $A_1, A_2, ..., A_n$. Then the measure of uncertainty about S, denoted by $H(S)$, can be called the entropy of the experiment S (properly called the Information Entropy).

1. Given that the entropy is

$$H = \sum_{i=1}^{n} -p_i \log(p_i) \tag{4}$$

and

$$\sum_{i=1}^{n} p_i = 1 \tag{5}$$

where $p_i = P(A_i)$ is the probability of event A_i
Prove that the entropy is maximized when **[5]**

$$p_1 = p_2 = ... = p_i = ... = p_n = \frac{1}{n} \tag{6}$$

Hint: You can consider using Mathematical Induction. But note that you can use any other method.

2. If $P(A_i|A_k) = \frac{P(A_iA_k)}{P(A_k)}$ is the conditional probability of event A_i given that the event A_k occurs. (where $P(A_iA_k)$ is the occurence of event A_i and A_k together). In other words, if the probability of occurrence of event A_i when A_k occurs is denoted by $P(A_i|A_k)$,

   (a) $P(A_i|A_k)_{k \neq i}$ is 0 for our system S. Why? [**0.5**]

   (b) Show that the uncertainity (entropy) about experiment S given that the event A_k occurs is also 0. [**1.5**]

   (c) Under what conditions would it not be zero? Why? [**0.5**]